Math 53: Multivariable Calculus **Sections 212, 213**

Worksheet for 2020-08-31

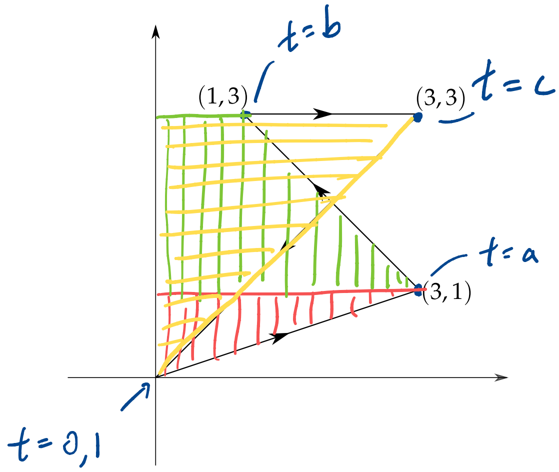
Questions marked with ** are less relevant to the core material and/or more difficult.

Problem 1. A particle starts at the origin at time $t = 0$ and follows the path $x = f(t)$, $y = g(t)$ illustrated in Figure 1. At time $t = 1$, it returns to the origin. Compute

$$\int_0^1 f(t)g'(t)\,\mathrm{d}t \text{ and } \int_0^1 g(t)f'(t)\,\mathrm{d}t$$

by interpreting them in terms of areas. How are the two values related to each other?

**It turns out that this relationship between these two integrals holds *as long as the path ends where it starts* (as it does in this example). Can you explain why?



$0<a<b<c<1$

FIGURE 1. Problem 1

$$\int_0^1 f(t)g'(t)\,dt = \underbrace{\int_0^a f(t)g'(t)\,dt}_{\int_0^1 x\,dy} + \underbrace{\int_a^b f(t)g'(t)\,dt}_{\int_1^3 x\,dy} + \underbrace{\int_b^c f(t)g'(t)\,dt}_{\cancel{\int_3^3 x\,dy} \to 0} + \underbrace{\int_c^1 f(t)g'(t)\,dt}_{\int_3^0 x\,dy}$$

=  $= 3/2$

=  $= 4$

= − $= -9/2$

So integral is $3/2 + 4 - 9/2 = \boxed{1}$.

By similar reasoning, can find $\int_0^1 g(t)f'(t)\,dt = \boxed{-1}$

Observe:
$$\int_0^1 f(t)g'(t)\,dt + \int_0^1 g(t)f'(t)\,dt$$

$$= \int_0^1 \frac{d}{dt}\left(f(t)g(t)\right)dt$$

$$= f(1)g(1) - f(0)g(0)$$

$$= 0.$$

(The curve ends where it started: $f(0) = f(1)$ and $g(0) = g(1)$.)

Problem 2. Find a Cartesian equation for the parametric curve $x = t^3 + t$, $y = t^2 + 2$. Then compute $\mathrm{d}y/\mathrm{d}x$, using (a) methods from Chapter 10, and (b**) implicit differentiation (hopefully at least one person in your group remembers how to do this!). Do you get the same answer?

From $y = t^2 + 2$ we get $t^2 = y - 2$.

Then $x = t(t^2+1)$

$$x^2 = t^2(t^2+1)^2 = (y-2)(y-1)^2 = y^3 - 4y^2 + 5y - 2$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2+1}$$

By implicit diff: $2x = 3y^2 \frac{dy}{dx} - 8y\frac{dy}{dx} + 5\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{2x}{3y^2 - 8y + 5}.$$

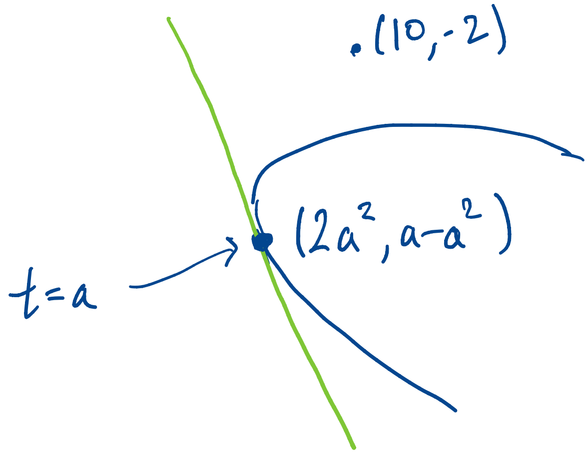
Looks different at a glance. But: $\dfrac{2x}{3y^2 - 8y + 5} = \dfrac{2t^3 + 2t}{3t^4 + 4t^2 + 1}$

$= \dfrac{2t(t^2+1)}{(t^2+1)(3t^2+1)} = \dfrac{2t}{3t^2+1}$ so it is actually the same.

Problem 3. There are two points on the curve

$$x = 2t^2,\ y = t - t^2,\ -\infty < t < \infty$$

where the tangent line passes through the point $(10, -2)$. Find these two points.



(figure not to scale)

At $t=a$, the slope of the tangent line is $\dfrac{1-2a}{4a}$ (if $a \neq 0$)

So the equation (by point-slope form) is

$$4a\left(y-(a-a^2)\right) = (1-2a)\left(x-2a^2\right)$$

(Note that this is valid even when $a=0$.)

We need to pass through $(10,-2)$. So:

$$4a\left(-2-(a-a^2)\right) = (1-2a)\left(10-2a^2\right).$$

Simplify and solve: $a = 1, 5$. So $(2a^2, a-a^2) = \boxed{(2,0) \text{ and } (50,-20)}$

are the points in question

Problem 4.

(a) Write down a parametrization $x = f(t), y = g(t)$ for the circle $x^2 + y^2 = 1$ which starts at $(1, 0)$ when $t = 0$, goes around once counterclockwise, and ends at $(1, 0)$ when $t = 2\pi$.

(b) Adjust your parametrization so that it completes the revolution in 1 unit of time, instead of 2π (i.e. it returns to $(1, 0)$ at $t = 1$ instead of $t = 2\pi$). Note that you are still parametrizing $x^2 + y^2 = 1$; the circle itself has not changed.

(c) Take this curve (the unit circle centered at the origin) and stretch it horizontally by a factor of 3 (so you get an ellipse centered at the origin). Write down a Cartesian equation and a parametrization for the result.

(d) Take the ellipse from (c) and translate it up in the positive y-direction by 5 units (so you get an ellipse centered at $(0, 5)$). Write down a Cartesian equation and a parametrization for the result.

(a) $x = \cos t$ $y = \sin t$ is the "standard" way.

(b) $x = \cos(2\pi t)$ $y = \sin(2\pi t)$.

(c) $\left(\frac{x}{3}\right)^2 + y^2 = 1$ $x = 3\cos t$ $y = \sin t$

(d) $\left(\frac{x}{3}\right)^2 + (y-5)^2 = 1$ $x = 3\cos t$ $y = \sin t + 5$

Problem 5** (Stereographic projection from the "north pole"). In Figure 2, the circle $x^2 + y^2 = 1$ has been depicted, together with a line passing through the points $(0, 1)$ and $(t, 0)$. This line intersects the circle at a point (other than $(0, 1)$), whose coordinates depend on the value of t. Find these coordinates $(f(t), g(t))$. Does the parametrization $x = f(t)$, $y = g(t)$, $-\infty < t < \infty$ trace out the entire unit circle?

See also: Stewart 10.1.40-44, which are similar in flavor (producing a parametrization from a geometric construction).

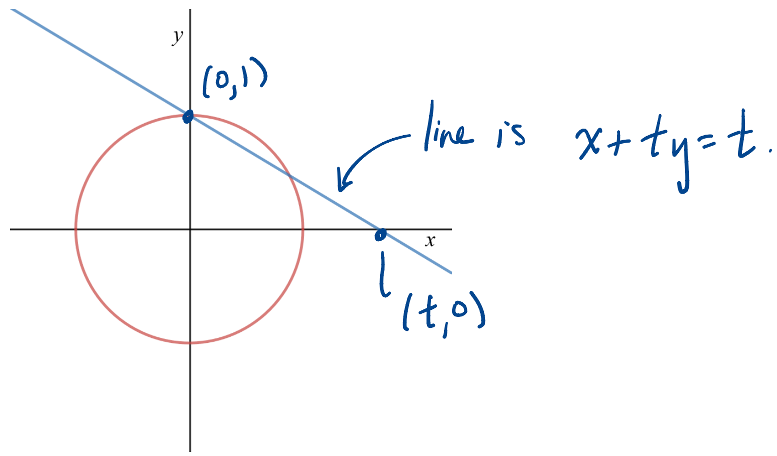


Figure 2. Problem 5

$ay^2+by+c=0$

$$\begin{cases} x^2+y^2=1 \\ x=t-ty \end{cases} \xrightarrow{\text{algebra...}} \underbrace{(t^2+1)}_{a}y^2 \underbrace{-2t^2}_{b}y+\underbrace{(t^2-1)}_{c}=0$$

We know $y=1$ is one solution (since the curves intersect at $(0,1)$). The other solution is thus

$$y=\frac{t^2-1}{t^2+1}$$ (b/c $-b/a$ is the sum of the roots, or c/a is the product)

So $x=t-ty=\dfrac{2t}{t^2+1}$.

(Check that $x^2+y^2=1$ if you want to make sure it works!)